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ON ONE MECHANISM OF FORMATION OF TORNADO-LIKE VORTICES IN A ROTATING FLUID

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It is shown that during excitation of forced, resonant, inertial oscillations of large amplitude in a rigidly rotating fluid, the mechanism of formation of tornado-like vortices is primarily of a kinematic nature (advection of circulation of the azimuthal component velocity and stretching of vortex lines by the poloidal components of the velocity field that arise from excitation of inertial oscillations). The main parameters of the vortices are obtained by solutions of model problems. To excite such oscillations, it is necessary to deliver energy far exceeding the initial energy of the rotating fluid. Therefore, inertial oscillations by themselves cannot lead to the occurrence of intense atmospheric vortices. Nevertheless, such oscillations can apparently play the role of a trigger mechanism that activates more complex processes of vortex formation related to instability of the atmosphere.

It has been established experimentally [1–3] that concentrated tornado-like vortices occur in a rigidly rotating fluid which entirely fills a cylindrical tank when the fluid is subjected to forced, resonant, inertial oscillations of large amplitude. In [1, 2], it was suggested that this phenomenon can serve as a model for intense atmospheric vortices (tornados, waterspout, etc.) and inertial oscillations play a leading role in their formation mechanism.

Confirmation (or refutation) of this hypothesis is an important step on the way to a deeper understanding of the indicated phenomena. In this connection, additional experiments [3] were performed on a setup whose parameters differed from the parameters of the setup described in [1, 2].

In the present work, we undertook an attempt to estimate the main characteristics of vortices produced by excitation of forced, resonant, inertial oscillations of large amplitude in a rigidly rotating fluid. The objective of these studies is to reveal the conditions and mechanism of formation of concentrated tornado-like vortices in a laboratory model, to determine their basic parameters and to estimate the importance of this phenomenon and its role in the formation of intense atmospheric vortices.

The main goal of the present study is to explain the experimentally observed increase in the concentration of vorticity in the neighborhood of the cylinder axis. In the core of a tornado-like vortex produced by excitation of oscillations, the vorticity amplitude is $\omega_z \simeq 50\omega_0$ (ω_0 is the vorticity of the unperturbed fluid which rotates at an angular velocity $\Omega = \omega_0/2$). The radius of the core is $r_0 \simeq 0.05R$ (R is the radius of the rotating cylinder) [1, 2]. There are no explanations for these experimental facts.

The available experimental data are rather limited. The maximum value of the azimuthal velocity and the radius at which this value is reached were measured in [1, 2]. An attempt to measure the azimuthal velocity field was undertaken in [3]. However, the scatter in the measurement data is so large that it is even impossible to establish whether the extreme value of the azimuthal velocity is reached in the region where the measurements were performed (in a coordinate system rotating together with the cylinder, this maximum is observed). This is explained by difficulties in carrying out such measurements and the nature of the phenomenon (irregular departures of the vortex from the axis of the cylinder). In this connection, it is of interest to estimate the main parameters of the vortices considered by solving model problems using experimental data. Such a rough approach is justified because it is presently unclear how viscosity, phenomena related to possible separation of the boundary layer, etc., are

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essential to the understanding of the formation mechanism of the observed vortices. Before trying to solve complex problems, it is necessary to determine whether it is possible in principle to explain the observed phenomenon using the model of an inviscid fluid.

Formulation of the Problem. Within the framework of the model of an incompressible (of unit density) viscous fluid, results of the experiment should be described by the solution of the problem formulated below. A cylinder of height H and radius R rotates at an angular velocity $\Omega = \omega_0/2$ around the z axis [$\omega_0 = \omega_z(0)$ is the initial vorticity of the fluid which rotates together with the cylinder]. It is required to determine oscillatory (periodic in time) fluid flows in the cylinder provided that one of the butt-ends of the cylinder is pliable and is deformed under the law $\boldsymbol{r} = (r = r(\rho,t), \varphi, z = z(\rho,t))$; in the undeformed state, $r(\rho,0) = \rho$ and $z(\rho,0) = 0$ (the z axis coincides with the axis of the cylinder), so that

$$z = h(r,t) < H \quad [z(R,t) = h(R,t) = 0], \qquad 2\pi \int\limits_0^R h(r,t) r\,dr = 0.$$

In this case, $h(r,t+T) = h(r,t)$, where T is the period, and $\omega = 2\pi/T$ is the frequency of the fundamental harmonic of the periodic perturbation.

Under the conditions considered, the constitutive equations describing nonstationary axisymmetric fluid flows can be written as follows [using cylindrical coordinates $\boldsymbol{r} = (r,\varphi,z)$ and $\boldsymbol{v} = (u,v,w)$]:

$$\Gamma_t = (\psi_z\Gamma_r - \psi_r\Gamma_z)/r + \nu(\Gamma_{rr} - \Gamma_r/r + \Gamma_{zz}); \tag{1}$$

$$\omega_t = \psi_z \frac{\partial}{\partial r}\Big(\frac{\omega}{r}\Big) - \psi_r \frac{\partial}{\partial z}\Big(\frac{\omega}{r}\Big) + \frac{\partial}{\partial z}\Big(\frac{\Gamma^2}{r^3}\Big) + \nu\Big(\omega_{rr} + \frac{1}{r}\,\omega_r - \frac{\omega}{r^2} + \omega_{zz}\Big) + f(r,z,t); \tag{2}$$

$$\psi_{rr} - \frac{1}{r}\,\psi_r + \psi_{zz} = -r\omega, \qquad u = -\frac{1}{r}\,\frac{\partial\psi}{\partial z}, \qquad w = \frac{1}{r}\,\frac{\partial\psi}{\partial r}, \qquad \omega_z = \frac{1}{r}\,\Gamma_r. \tag{3}$$

Here $\omega = \omega_\varphi = u_z - w_r$, $\Gamma = rv$, $f(r,z,t) = \boldsymbol{e}_\varphi \operatorname{rot} \boldsymbol{F}$ [$\boldsymbol{F}(r,z,t)$ is the poloidal vector of external mass forces (in the experiment described above, such forces are absent)]. The addition of these forces allows one to model forced oscillations without using the pliable butt-end of the cylinder, which simplifies the problem.

The solutions of system (1)–(3) should satisfy the following boundary conditions:

— $\psi = 0$ at $r = 0$, $r = R$, and $z = H$;

— $u = 0$ and $w_r = 0$ at $r = 0$, $u = 0$ at $r = R$, and $w = 0$ at $z = H$;

— $h_t = w + uh_r$, $r_t = u$, and $z_t = w$ at $z = h(r,t)$ [($r = r(\rho,t)$, $z = z(\rho,t)$ are specified functions, whose form is determined by the form of the strain (bending and tension) of the butt-end surface].

In this formulation, the problem is very complicated. In fact, of interest here are not details of the motion (rather complicated) but only an answer to the question of whether vorticity concentration is possible or not during excitation of poloidal oscillatory motion and estimates of the parameters of such concentration?

The problem is simplified if viscosity is ignored. In this case, the solutions of system (1)–(3) ($\nu = 0$) should satisfy the following conditions:

— $\psi = 0$ at $r = 0$ and $r = R$ (and $z = 0$ and $z = H$ if the butt-end of the cylinder is not deformed);

— $\psi = 0$ at $r = 0$, $r = R$, and $z = H$; $h_t = w + uh_r$ at $z = h(r,t)$.

In the first case, it is possible to formulate the problem of free oscillations [if $f(r,z,t) = 0$] and forced oscillations [if $f(r,z,t) \neq 0$]. However, in this case, too, the problem remains extremely complicated.

There have been a great number of papers in which perturbations in a rigidly rotating, incompressible fluid were studied in an inviscid linear formulation. The problem is nontrivial even in this case. The spectrum of free oscillations was found to have a complex structure [4, 5]. Of interest for the problem considered here is a paper [4], in which forced oscillations of a fluid in a rotating cylinder with a pliable cover were studied in a linear formulation (with initial conditions) and a rather complicated structure of the flow was determined.

The large oscillation amplitude is extremely essential to an understanding of the phenomenon considered, which requires solving the corresponding problem in a nonlinear formulation. Studies of the present problem in this formulation have not been reported in the literature.

Nonlinear oscillations have a number of special features which differ them from linear oscillations. This is demonstrated by a simple model of a nonlinear oscillating system — the Duffing equation [6, 7]. A characteristic feature of free nonlinear oscillations is the dependence of the period of oscillations on amplitude. In the case of forced oscillations there are nonuniqueness (for steady-state oscillations) and chaotic behavior if the problem is solved subject to initial conditions. All these features are inherent in solutions of ordinary differential equations. In particular, it is known that for some values of the parameters included in these equations, forced oscillations become random [6]. It is unlikely that the nonlinear behavior of distributed-parameter systems is simpler. Various approximate and numerical methods of studying nonlinear systems are discussed in [6, 7] and others. From these studies, it can be concluded that the numerical calculation of nonlinear problems with periodic conditions in time (for a continuous medium) is very complicated and requires an in-depth preliminary mathematical analysis and powerful computing resources.

In connection with the aforesaid, it is of interest to study much simpler model formulations which take into account the action of poloidal flow on the evolution of circulation and vertical vorticity. This allows one to determine conditions for concentration of vertical vorticity and to estimate the degree (dimension of the region) of this concentration.

The following formulations are of interest:

(a) A function $f(t, r, z)$ is specified and the forced steady-stated oscillations induced by time-periodic mass forces are determined;

(b) A time-periodic stream function $\psi(t, r, z)$ is specified and Eq. (1) is solved. Then, Γ, ω, ω_z, and the radius of the zone of concentration of the vertical vorticity component are determined. The distribution of the mass forces $f(t, r, z)$ is determined from Eq. (2) (in a sense, the inverse problem).

In the present paper, we consider the simplest version (b).

Model Formulation. In a linear approximation, the stream function of steady-state forced oscillations in the formulation considered is determined by the infinite series

$$\psi = \frac{A_1}{\sin k_1\pi}\, rJ_1\Big(\frac{\mu_1 r}{R}\Big)\sin\Big(k_1\pi\Big(1-\frac{z}{H}\Big)\Big) + \sum_{n=2}^{\infty}\frac{A_n}{\sin k_n\pi}\, rJ_1\Big(\frac{\mu_n r}{R}\Big)\sin\Big(k_n\pi\Big(1-\frac{z}{H}\Big)\Big), \qquad (4)$$

where the constant coefficients A_n are determined from the boundary condition $h_t = \psi_r/r$ at $z = 0$. The quantities μ_n are roots of the equations $J_n(\mu_n) = 0$, and the quantities k_n are defined by the formula

$$k_n = \mu_n H\omega_*/(\pi R\sqrt{\omega_0^2 - \omega_*^2}\,).$$

If the forced oscillation frequency ω_* is close to one of the resonant frequencies ω_n, then the corresponding quantity k_n differs little from the integer n, and the amplitude of this mode is large. In the experiments of [1–3], resonant inertial oscillations were excited with the frequency corresponding to the fundamental (first) mode, which in (4) was extracted from the general series. If forced oscillations are excited from the state of rigid rotation (as in the indicated experiments), all modes are excited but the amplitude of the resonant mode increases and becomes dominant after a time. Actually, the process is more complicated (see [4]) but for the problem considered, this circumstance is not so important. However, when the oscillation amplitude becomes large enough, the linear approximation is no longer valid and the flow is not described by the resonant mode of series (4). Nevertheless, results of experiments show that the flow pattern (topology of streamlines) does not change when oscillations are excited at the fundamental frequency even with very large amplitudes. Bearing this in mind, as trial periodic (in time) stream functions, we use the stream functions corresponding to the resonant mode and close to it, ignoring the curvature of the butt-end surface used to excite oscillations. In this approach, the oscillation amplitude remains undetermined (arbitrary), and its value is determined from a comparison of calculations with experimental results. Thus, from the aforesaid, the choice of a particular type of stream function is determined by its proximity to the eigenfunction of the fundamental mode of inertial oscillations in the cylinder (with zero boundary conditions), the requirement $\psi \to r^2$ as $r \to 0$, and the possibility of obtaining analytical results even if in some cases.

Below, we use dimensionless (marked by asterisks) variables related to the initial ones as follows:

$$r = \frac{R}{\pi}r_*, \quad z = \frac{H}{\pi}z_*, \quad t = \frac{T}{2\pi}\,t_*, \quad u = \frac{2R}{T}\,u_*, \quad w = \frac{2H}{T}\,w_*, \quad v = \frac{1}{2}\,\omega_0 R v_*,$$

$$\psi = \frac{2HR^2}{\pi^2 T}\,\psi_*, \qquad \Gamma = \frac{\omega_0 R^2}{\pi^2}\,\Gamma_*, \qquad \omega_z = \omega_0\omega_{z*}.$$

Further, the asterisks are omitted.

Let a time-periodic stream function be specified. For example,

$$\psi = A\ \Psi(r)\sin z \sin t,$$

where for $\Psi(r)$, two versions are considered:

$$\text{No. 1:}\quad \Psi_1(r) = rJ_1(\mu_1 r/\pi), \qquad \text{No. 2:}\quad \Psi_2(r) = r\sin r.$$

These versions are close to each other but version No. 2 allows the result to be obtained in analytic form [distribution of $v(r)$ at the butt-ends of the cylinder].

If the stream function is specified, determination of the azimuthal velocity $v(r,z,t)$ and the vertical vorticity $\omega_z(r,z,t)$ reduces to solving Eq. (1) for $\Gamma(r,z,t)$, in which it is necessary to set $\nu = 0$. We have

$$\Gamma_t + u\Gamma_r + w\Gamma_z = 0.$$

The motion of the fluid particles is described by the equations

$$\frac{dr}{dt} = u = -\frac{1}{r}\,\psi_z, \qquad \frac{dz}{dt} = w = \frac{1}{r}\,\psi_r. \tag{5}$$

The solution of this system with the initial conditions $r = x$ and $z = y$ at $t = 0$ determines the solution of the equation for Γ, because according to this equation, the quantity Γ remains constant during motion along the trajectory $r = r(t,x)$, $z = z(t,y)$; at $t = 0$, we have $\Gamma = x^2/2$. A complete analytical solution of this problem was not obtained. However, such solutions can be obtained in both versions (Nos. 1 and 2) for particles located on the cylinder axis (and these solutions coincide), and for the fluid particles at its butt-ends, it can be obtained in version No. 2. We consider the motion of particles on the cylinder axis. Here $r = 0$, and for $z = z(t,y)$, we have the equation

$$\frac{dz}{dt} = 2A\sin t\sin z, \qquad t = 0, \quad z = y.$$

The solution has the form

$$z = 2\arctan\,[\exp(2\varphi(t))\tan(y/2)], \tag{6}$$

where $\varphi(t) = A(1-\cos t)$.

From (6), we obtain

$$z_y = \frac{\exp(2\varphi)}{(1+\exp(4\varphi)\tan^2(y/2))\cos^2(y/2)} = \frac{\exp(2\varphi)}{1+(\exp(4\varphi)-1)\sin^2(y/2)}.$$

Hence, taking into account that $\omega_z(y,t) = z_y$ and $\tan(y/2) = \exp(-2\varphi(t))\tan(z/2)$, we obtain

$$\omega_z(z,t) = \exp(2\varphi(t))\cos^2(z/2) + \exp(-2\varphi(t))\sin^2(z/2).$$

For the phase of maximum (at the lower butt-end) vorticity, we have

$$z = 0: \qquad \omega_z(0,t) = \exp(4A), \quad \varphi(\pi) = 2A,$$

$$z = H/2: \qquad \omega_z(H/2,t) = (1/2)\cosh 4A,$$

$$z = H: \qquad \omega_z(H,t) = \exp(-4A).$$

The quantity A is chosen from a comparison of calculation results with experimental data. For $A = 0.4$–1.0, we have $\omega_z(0,t) = \exp(2\varphi(t)_{\max}) \simeq 5$–$50$, which corresponds to the range of this quantity in experiments. The value $A \simeq 1$ corresponds to the maximum (for vorticity) values obtained in [1, 2], and $A \simeq 0.44$ corresponds to the maximum values obtained in [3]. We note that in the phase of maximum vorticity at the lower butt-end, $\omega_z(H/2) \simeq \omega_z(0)/2$ (if $A \simeq 1$), which is close to the results of [1, 2], where $\omega_z(H/2) \simeq 0.4\omega_z(0)$ is obtained.

Let us consider the motion at the lower ($z = 0$) butt-end, using the stream function $\Psi_2(r)$. Here

$$\frac{dr}{dt} = -A\sin t\sin r, \qquad t = 0, \quad r = x.$$

The solution of this equation has the form

$$r(t,x) = 2\arctan\,(\exp(-\varphi)\tan\,(x/2)).$$

At $t = 0$, we have $\Gamma(x) = x^2/2$. Hence,

$$\Gamma(t, r) = 2[\arctan(\exp(\varphi)\tan(r/2))]^2.$$

The azimuthal velocity component $v(r, t)$ is evaluated from the formula

$$v(r, t) = (2/(\pi r))\Gamma(r, t) = (2/(\pi r))[\arctan(\exp(\varphi)\tan(r/2))]^2.$$

The vorticity distribution $\omega(r) = \Gamma_r/r$ has the form

$$\omega(t, r) = 2\exp(\varphi)\,\frac{\arctan(\exp(\varphi)\tan(r/2))}{r(\sin^2(r/2) + \exp(\varphi)\cos^2(r/2))}.$$

From this it follows that for $A \simeq 1$, concentration of vorticity occurs in the region $r \sim 0.1R$. In this case, $v \sim 3V_0$ (in version No. 1, numerical calculations showed that for $A = 1$, $r = 0.1R$ and $v = 2.8V_0$), which is agrees qualitatively with measurements in [1, 2]. For $A = 0.44$, we obtain $r \simeq 0.16$ and $v \simeq 0.7$, which agrees qualitatively with results of [3] but this value of the azimuthal velocity is not extreme (in the neighborhood of this point, the maximum value is the azimuthal velocity in a coordinate system attached to the rotating cylinder).

Let us revert to system (4), (5) and determine the circulation distribution $\Gamma(r)$, $v(r)$, and $\omega(r)$ at $z = \pi/2$ in the phase of maximum vorticity concentration. The displacement of the point $(r = x,\ z = \pi/2)$ is defined by Eqs. (4) and (5) with the initial conditions $r = x$ and $z = \pi/2$ at $t = 0$. Integrating these equations to $t = \pi$, we find that $r_1 = r(x, \pi)$ and $z_1 = z(x, \pi)$. Apparently, a particle that was at the point $[r_1 = r(x, \pi),\ z_1 = -z(x, \pi)]$ at $t = 0$ will arrive at the point (x, π). Because circulation is conserved, the equality

$$\Gamma(x) = \Gamma(r_1) = r_1^2(x)/2$$

should hold. Hence (assuming $x = r$), we obtain the azimuthal velocity component

$$v(r) = r_1^2(r)/(\pi r).$$

The vorticity distribution is evaluated by the formula

$$\omega(r) = \frac{1}{r}\,\Gamma'(r) = \frac{1}{r}\,r_1\,\frac{dr_1}{dr}.$$

The procedure described above was implemented by numerical calculations. Results of one of the calculations (in version No. 2, for an initial height $H/2 = 19.3$ cm and $A = 0.7$) are shown in Fig. 1. The family of curves corresponds to particle trajectories. The ends of the trajectories determine the position of the particles (at $t = \pi$). The points correspond to values of the azimuthal velocity at height $H/2$ versus the radius in the phase of maximum vorticity. The maximum azimuthal velocity $v = 1.1V_0$ is reached at a radius $r = 6.5$ cm. It is evident that there is only qualitative agreement with measurement data.

Nevertheless, the results obtained suggest that the experimentally observed increase of vorticity and its concentration on the cylinder axis is primarily a kinematic effect (advection of circulation of the azimuthal velocity component and stretching of the vorticity lines by the poloidal components of the velocity field), which can be explained within the framework of the inviscid fluid model. To obtain further arguments in favor of this standpoint, below we consider a model problem taking into account viscosity.

Estimate of the Viscosity Effect. To estimate the effect of viscosity, we consider the following problem. System (1)–(3) have solutions of the form (below dimensional variables are used):

$$\psi(r, z, t) = f(t)r^2 z, \qquad \omega(r, z, t) = 0, \qquad \Gamma(r, z, t) = \Gamma(r, t).$$

Here $\Gamma(r, t)$ satisfies the equation

$$\Gamma_t - f(t)r\Gamma_r = \nu(\Gamma_{rr} - \Gamma_r/r). \tag{7}$$

Equation (7) has solutions of the form

$$\Gamma = \Gamma_0[1 - \exp(-\gamma(t)r^2)],$$

where $\gamma(t)$ satisfies the equation $\gamma' = 2f(t)\gamma - 4\nu\gamma^2$, which, in turn, has the solution

$$\gamma(t) = \gamma_0 \exp\left(2\int_0^t f(\tau)\,d\tau\right)\Bigg/\left[1 + 4\nu\gamma_0\int_0^t \exp\left(2\int_0^t f(\tau)\,d\tau\right)\right].$$

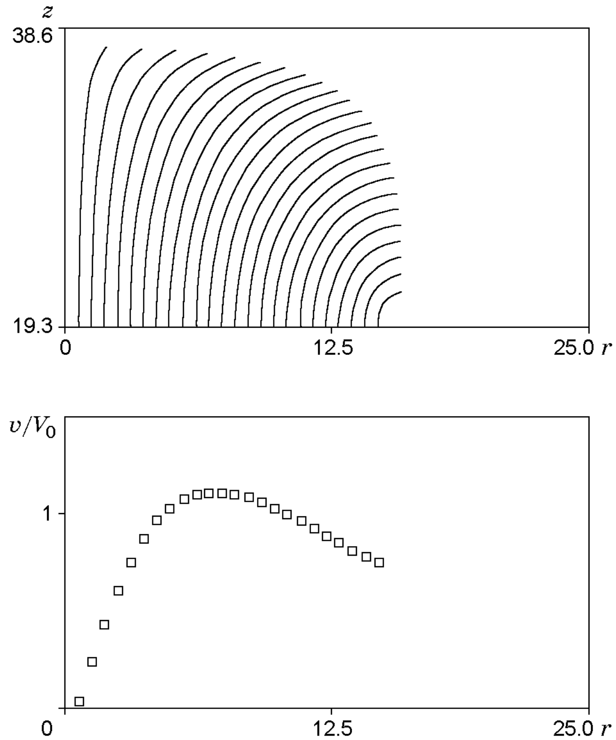

Fig. 1

Here $\gamma_0 = \gamma(0)$. Solutions of this form are obtained in [8]. Let us assume that at $t = 0$ there is a stationary solution — a Burgers vortex:

$$\Gamma = \Gamma_0[1 - \exp(-\gamma_0 r^2)], \quad u_0 = -\psi_z/r = -2\nu\gamma_0 r, \quad w_0 = \psi_r/r = 4\nu\gamma_0 z. \tag{8}$$

We assume that in this solution, the maximum of the azimuthal velocity $v = V_0$ is reached at $r = R$. Then, the solution can be written as

$$\Gamma = \frac{\Gamma_0}{1 - \exp(-\mu)}\Big(1 - \exp\Big(-\mu\,\frac{r^2}{R^2}\Big)\Big), \qquad \omega_z = \omega_0 \exp\Big(-\mu\,\frac{r^2}{R^2}\Big),$$

where $\Gamma_0 = V_0 R$, $\omega_0 = (\Gamma_0/R^2)\exp(-\mu)$, $V_0 = \exp(-\mu)\omega_0 R$, $\gamma_0 = \mu/R^2$, and $\mu = 1.25$ is a nonzero root of the equation $\exp(\mu) = 1 + 2\mu$.

In experiments whose results are given in [3], $R = 25$ cm, $V_0 = 65.4$ cm/sec, and $\nu = 0.01$ cm^2/sec. Then, for the steady-state flow at the radius R, we have $|u| = 2\mu\nu/R = 10^{-3}$ cm/sec.

At $t > 0$, let $f(t) = -2\nu\gamma_0 - AP(t/T)$, where $P(\tau) = P(\tau + 1)$ is a periodic function with period T, such that $\max|P(\tau)| = 1$ and $P(0) = 0$, $P'(0) > 0$, $\int_0^1 P(\tau)\,d\tau = 0$, and $\tau = t/T$, i.e., a periodic perturbation with the poloidal component of the velocity field is superimposed on the initial steady-state flow. Then, at $t > 0$, we have

$$u = -(2\nu\gamma_0 + AP(t/T))r, \qquad w = 2(2\nu\gamma_0 + AP(t/T))z.$$

This formulation simulates to some extent the flow at the fixed butt-end (ignoring the boundary layer due to viscous attachment to the butt-end) or (more precisely) the flow in the vicinity of the velocity node w if the resonant oscillation mode with wave length equal to the height of the cylinder is excited. It is of interest to study the problem of the effect of the viscosity on the magnitude of vorticity and the degree of its concentration.

If $\nu = 0$, the oscillating poloidal flow

$$u = -AP(t/T)r, \qquad w = 2AP(t/T)z$$

is superimposed on the same initial field of the azimuthal velocity (8) (for $\nu = 0$, the initial poloidal flow is absent: $u_0 = w_0 = 0$). In this case, the solution has the form

$$\Gamma(\tau) = \Gamma_0(1 - \exp(-\gamma(\tau) r^2)), \qquad \gamma(\tau) = \gamma_0 \exp\left(2AT \int_0^\tau P(\tau')\, d\tau'\right).$$

The vorticity on the axis of the vortex $r = 0$ is

$$\omega_z(\tau) = \omega_0 \gamma(\tau)/\gamma_0, \qquad \omega_0 = 2\gamma_0 \Gamma_0.$$

Hence,

$$\omega_z(\tau)/\omega_0 = \exp(\beta g(\tau)),$$

where $\beta = 2AT$ and $g(\tau) = \int_0^\tau P(\tau')\, d\tau'$ is the periodic function $g(\tau + 1) = g(\tau) > 0$, and $0 < g(\tau) < 1$.

In the case of a viscous liquid [$\gamma(0) = \gamma_0$], we obtain

$$\gamma(\tau) = \gamma_0 F(\tau) \bigg/ \left(1 + \alpha \int_0^\tau F(\tau')\, d\tau'\right), \qquad F(\tau) = \exp(\alpha\tau + \beta g(\tau))$$

for $\tau > 0$. Here $\alpha = 4\nu\gamma_0 T$.

It is easy to show that as $\tau \to \infty$, the function $\gamma(\tau)$ tends to a function $\gamma_*(\tau)$ such that $\gamma_*(\tau + 1) = \gamma_*(\tau)$, i.e., the flow enters a periodic regime with period T that coincides with the period of the external action, and

$$\gamma_*(\tau) = \gamma_* \gamma_0 F(\tau) \bigg/ \left(\gamma_0 + \alpha\gamma_* \int_0^\tau F(\tau')\, d\tau'\right), \qquad \gamma_* = \gamma_*(0) = (\exp(\alpha) - 1) \bigg/ \left(\int_0^1 F(\tau)\, d\tau\right).$$

The vorticity on the axis $r = 0$ is defined by the equality

$$\omega_z(\tau) = \omega_0 \gamma(\tau)/\gamma_0.$$

It can be shown that $\gamma(\tau)/\gamma_0 < \exp(\beta g(\tau))$, i.e., the maximum vorticity on the vortex axis is always smaller than that in an inviscid fluid. From this it follows that the degree of vorticity concentration is also lower than that in an ideal liquid. However, if $\alpha\tau \ll 1$, this difference is small, and it becomes significant only for $\alpha\tau > 1$, which corresponds to the condition $t > 10^3$ sec ($\tau > 400$). In this case, the vorticity on the axis fluctuates around a value that is little different from the initial one: $\omega_z(0, \tau)/\omega_0 = \gamma_*(\tau)/\gamma_0 \sim 1$. The time for transition of oscillations to this regime is large enough (approximately 20 min), and usually, this regime is not regularly observed in experiments (in addition, in a closed volume, the flow can evolve differently). However, it is improbable that a specially arranged experiment will confirm the possibility of existence of this regime.

Thus, in the volume away from the walls of the cylinder, the effect of viscosity is insignificant. At the butt-ends of the cylinder, the effect of Ekman and Stokes boundary layers (related to the radial flow) can be significant and requires further investigation.

Energy Estimate. As will be shown below, the energy of the flow of interest is far in excess (by severalfold) of the energy of the initial rigid rotation. From this it follows that the excited inertial oscillations by themselves cannot give rise to a tornado-like vortex — energy input is required. In experiments, such energy input is ensured by the work performed by the oscillating pliable butt-end of the cylinder. Under natural conditions where inertial oscillations arise from the interaction of a mesocyclone with the roughness of the earth's surface, the only possible source of additional energy can be the potential energy due to the unsteady stratification of the atmosphere. In this case, the processes related to convective instability play an important role, and inertial oscillations can be a trigger mechanism that initiates these processes.

Let us consider an idealized process in which an axisymmetric flow changes from rigid rotation to potentially rigid flow for some reasons (without specifying them). We assume conservation of the kinetic energy and the axial component of the momentum (as in the case of excitation of free inertial oscillations). The energy and momentum in the initial and final states are obtained assuming that the flow rearrangement results in a potentially rigid flow, in which at $r < r_*$, the fluid rotates rigidly, and at $r > r_*$, the flow is potential. Then, at the initial time, the momentum is

$$K_0 = (\pi\rho/2) V_0 R^3 H,$$

and the energy is

$$E_0 = (\pi\rho/4) V_0^2 R^2 H.$$

After the rearrangement of the momentum,

$$K_* = \pi\rho V_* r_* (R^2 - r_*^2/2)H,$$

the energy becomes

$$E_* = (\pi\rho/4)V_*^2 r_*^2 H(1 + 4\ln(R/r_*)).$$

By virtue of conservation of momentum and energy, the flow parameters V_* and r_* are obtained from the system

$$V_* r_* (2R^2 - r_*^2) = V_0 R^3, \qquad V_*^2 r_*^2 (1 + 4\ln(R/r_*)) = V_0^2 R^2.$$

From this we have $r_* \simeq 0.52R$, $V_* \simeq 1.14V_0$, and $\omega_* \simeq 2.1\omega_0$, which is apparently inconsistent with experimental results. In order that the values of the desired quantities be comparable with those obtained in experiments, additional (considerable) energy should be added to the right side of the energy equation. In the experiment with excitation of inertial oscillations, energy supply is accomplished by the work performed by the deformation of the pliable butt-end cover of the rotating cylinder.

Under natural conditions, additional energy can originate only from the potential energy of the unsteadily stratified atmosphere. The quantity of this energy can be estimated as follows. We assume that the atmosphere is unsteadily (linearly) stratified: $\rho = \rho_0 + \Delta\rho z/H$. In the state of steady equilibrium, $\rho = \rho_0 + \Delta\rho - \Delta\rho z/H$ because the total mass of the fluid is the same in both cases. Then, the content of potential energy is calculated from the formula

$$E_p = \int\limits_0^H \Big(\rho_0 + \Delta\rho\,\frac{z}{H}\Big) gz\,dz - \int\limits_0^H \Big(\rho_0 + \Delta\rho - \Delta\rho\,\frac{z}{H}\Big) gz\,dz = \frac{1}{6}\,\Delta\rho g H^2.$$

At $H \sim 1$ km, this quantity can be higher by severalfold than the kinetic energy of the mesocyclon from which a tornado originates. Convective flows play an important role in such energy supply, and it would be wrong to speak about the origin of a tornado-like vortex only due to excitation of inertial oscillations. However, the vorticity concentration achievable only due to excitation of inertial oscillations can serve as a trigger mechanism for development of instability and conversion of the potential energy of the unsteadily stratified atmosphere into the kinetic energy of a tornado.

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